Readings/problems 4 for Physics 250, due 3/12/09

Readings: Read set 3 of lecture notes; read tutorial by Ong et al. on Berry-phase theory of AHE. Read an introductory article on fractional quantum Hall effect (e.g., in volume edited by Prange and Girvin).

Credit: Students receiving 2 units should solve any 2 problems.

1. (a) Explain why the Berry curvature in a Bloch electron system vanishes everywhere in k-space in systems that have both inversion and time-reversal symmetry. What does this say about the anomalous Hall effect in such a system? (b) Construct a 2D band structure with two nondegenerate bands and Chern number 1 for the lower band.

Answer: (a) The Berry curvature is even under inversion symmetry and odd under time-reversal, and hence must vanish in systems with both symmetries. Hence the anomalous Hall effect is zero; actually, because of the integration over the whole band, the total DC AHE will vanish if time-reversal is present, whatever happens with inversion.

(b) We want to find a map from the torus to the sphere that covers the sphere exactly once. To do this, we can use the a slightly modified version of the mapping described in problem 2 of problem set 3: let each constant k_x circle, parametrized by k_y, pass through the north pole and the point $\theta = k_x + \pi$ along some fixed line of longitude (here $\theta > \pi$ gives the point diametrically opposite $\theta - \pi$). This covers the sphere once and is periodic in k_x and k_y.

2. Construct a 1D tight-binding model with two inequivalent sites per unit cell and a nonzero electrical polarization from the Berry-phase formula. Define a periodic parameter λ in your model that will pump a single electronic charge when changed from 0 to 2π.

Answer: Actually we can reinterpret the answer of 1(b). The two-dimensional Hamiltonian there with one occupied state now describes the two sites in the unit cell. There $H(k_x, k_y)$ has Chern number 1. Treating k_y now as the external parameter λ, the Chern number becomes the charge pumped. At general values of λ, the polarization is nonzero.

3. Exercises on spin chains: (a) Take three spin-half sites in a triangle with equal antiferromagnetic exchange couplings, i.e.,

$$H = J \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j. \tag{1}$$

What is the ground state energy of this three-site Heisenberg model? What is the ground state of the XX model (i.e., leaving out the s^z coupling)? (b) Consider two sites with $s = 1$ spins. What is the ground state energy of the above Heisenberg interaction? Consider an additional biquadratic interaction $\beta J(\mathbf{s}_i \cdot \mathbf{s}_j)^2$. Compute the ground state energy of the two spins with this additional interaction as a function of β. Explain why such an interaction is redundant for $s = 1/2$.

Answer: (a) First write H as

$$H = \frac{J}{2}\left[\left(\sum_i \mathbf{s}_i\right)^2 - {s_1}^2 - {s_2}^2 - {s_3}^2\right] = \begin{cases} \frac{J\hbar^2}{2}((3/2)(5/2) - 3(3/4)) & \text{if total spin } 3/2 \\ \frac{J\hbar^2}{2}((1/2)(3/2) - 3(3/4)) & \text{if total spin } 1/2. \end{cases} \tag{2}$$

The second case gives a four-fold degenerate ground state of energy $E_0 = -3J\hbar^2/4$, while the first gives a multiplet of four excited states (since three spins form two spin-half multiplets plus one s=3/2 multiplet) at $E_1 = 3J\hbar^2/4$.

For the XX case, we start with the above states and subtract the energy

$$H' = \frac{J}{2}\left[\left(\sum_i s_i^z\right)^2 - {s_1^z}^2 - {s_2^z}^2 - {s_3^z}^2\right] \tag{3}$$

Each s_z^2 just gives $\hbar^2/4$, while the first term gives different answers depending on the total s_z state. However, all four ground states have total s_z equal to $\pm\hbar/2$, so the net effect of this term is just to add $J\hbar^2/4$ to the above ground state energy. For the four previous excited states, two move up in energy by this amount, but the two with total s_z equal to $\pm 3\hbar/2$ move down by $3J\hbar^2/4$. So there are now 2 states at $J\hbar^2$, 4 states at $-J\hbar^2/2$, and 2 states at 0.

(b) The two spins can form total spin 0, 1, or 2 multiplets, with energy $-2J\hbar^2$ for the first case, $-J\hbar^2$ for the second, and $J\hbar^2$ for the third. To treat the biquadratic interaction, we note that the above states are also eigenstates of the biquadratic interaction. The energies are then

$$-2J\hbar^2 + \beta J(2\hbar^2)^2, \quad -J\hbar^2 + \beta J(\hbar^2)^2, \quad J\hbar^2 + \beta J(\hbar^2)^2. \tag{4}$$

For small β, the first is smallest and the energy is $J\hbar^2(4\beta\hbar^2 - 2)$. For large β, the second is smallest and the energy is $J\hbar^2(\beta\hbar^2 - 1)$. The two are degenerate at $\beta = 1/3$, as we used in constructing the AKLT Hamiltonian. This interaction is redundant for $s = 1/2$ because it can be expanded over the identity and the ordinary quadratic interaction.

4. Compute the normalized lowest Landau level eigenfunctions of a two-dimensional electron in a rotationally symmetric gauge $\mathbf{A} = (-By/2, Bx/2, 0)$. Check that the areal density is as expected: $n = \frac{1}{2\pi\ell^2}$, ℓ the magnetic length. Hint: exploit rotational symmetry and note that these eigenfunctions take an especially simple form in complex notation $z = x + iy$.

Answer: The eigenfunctions, as we have used in class, are found to be of the form $C_m z^m \exp(-|z|^2/4\ell^2)$, where $\ell = \sqrt{\hbar c/eB}$ is the magnetic length. The normalization constant C_m is determined by

$$1 = |C_m|^2(2\pi)\int_0^\infty r\,dr\,r^{2m}\exp(-r^2/2\ell^2) = |C_m|^2\ell^{2m+1}(2\pi)\Gamma(m+1)2^m. \tag{5}$$

so

$$C_m = \frac{1}{\sqrt{2^m\Gamma(m+1)\ell^{2m+1}}} \tag{6}$$