

1) **Old Exam problem:** A two-form is expressed in Cartesian coordinates as,

$$\omega = \frac{1}{r^3}(zdx dy + xdy dz + ydz dx)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

- a) Evaluate $d\omega$ for $r \neq 0$.
- b) Evaluate the integral

$$\Phi = \int_P \omega$$

over the infinite plane $P = \{-\infty < x < \infty, -\infty < y < \infty, z = 1\}$.

- c) A sphere is embedded into \mathbb{R}^3 by the map φ , which takes the point $(\theta, \phi) \in S^2$ to the point $(x, y, z) \in \mathbb{R}^3$, where

$$\begin{aligned}x &= R \cos \phi \sin \theta \\y &= R \sin \phi \sin \theta \\z &= R \cos \theta.\end{aligned}$$

Pull back ω and find the 2-form $\varphi^*\omega$ on the sphere. (**Hint:** The form $\varphi^*\omega$ is both familiar and simple. If you end up with an intractable mess of trigonometric functions, you have made an algebraic error.)

- d) By exploiting the result of part c), or otherwise, evaluate the integral

$$\Phi = \int_{S^2(R)} \omega$$

where $S^2(R)$ is the surface of a two-sphere of radius R centered at the origin.

2) **Sphere Area:** The sphere S^n can be thought of as the locus of points in \mathbb{R}^{n+1} obeying $\sum_{i=1}^{n+1} (x^i)^2 = 1$. Use its invariance under orthogonal transformations to show that the element of surface “volume” of the n -sphere can be written as

$$d(\text{Volume on } S^n) = \frac{1}{n!} \epsilon_{\alpha_1 \alpha_2 \dots \alpha_{n+1}} x^{\alpha_1} dx^{\alpha_2} \dots dx^{\alpha_{n+1}}.$$

Use Stokes’ theorem to relate the integral of this form over the *surface* of the sphere to the volume of the *solid* unit sphere. Confirm that we get the correct proportionality between the volume of the solid unit sphere and the volume or area of its surface.

3) Push and Pull: Use a local co-ordinate system to work the following exercises:

a) Show that the operation of taking an exterior derivative commutes with a pull back:

$$d[\phi^*\omega] = \phi^*(d\omega), \quad \omega \in \bigwedge^p T^*N.$$

b) If the map $\phi : M \rightarrow N$ is invertible then we may push forward a vector field X on M to get a vector field ϕ_*X on N . Show that

$$\mathcal{L}_X[\phi^*\omega] = \phi^*[\mathcal{L}_{\phi_*X}\omega], \quad \omega \in \bigwedge^p T^*N.$$

c) Again assume that $\phi : M \rightarrow N$ is invertible. By using the co-ordinate expressions for the Lie bracket and the effect of a push-forward, show that if X, Y are vector fields on TM then

$$\phi_*([X, Y]) = [\phi_*X, \phi_*Y],$$

as vector fields on TN .

4) Sterographic Co-ordinates: The stereographic map $S^2 \rightarrow \mathbb{C}$ takes the point on S^2 with spherical polar co-ordinates θ, ϕ to the complex number

$$\zeta = e^{i\phi} \tan \theta/2.$$

We can therefore set $\zeta = \xi + i\eta$ and take ζ, η as defining a *stereographic co-ordinate system* on the sphere. Show that in these co-ordinates the sphere metric is given by

$$\begin{aligned} g(,) &\equiv d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi \\ &= \frac{2}{(1 + |\zeta|^2)^2} (d\bar{\zeta} \otimes d\zeta + d\zeta \otimes d\bar{\zeta}) \\ &= \frac{4}{(1 + \xi^2 + \eta^2)^2} (d\xi \otimes d\xi + d\eta \otimes d\eta), \end{aligned}$$

and the area 2-form becomes

$$\begin{aligned} \Omega &\equiv \sin \theta d\theta \wedge d\phi \\ &= \frac{2i}{(1 + |\zeta|^2)^2} d\zeta \wedge d\bar{\zeta} \\ &= \frac{4}{(1 + \xi^2 + \eta^2)^2} d\xi \wedge d\eta. \end{aligned} \tag{1}$$

5) Bogomolnyi Equations In this problem you will find the spin field $n : x \mapsto \mathbf{n}(x)$ that minimizes the non-linear σ -model energy functional

$$E[\mathbf{n}] = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla n^1|^2 + |\nabla n^2|^2 + |\nabla n^3|^2) dx^1 dx^2$$

for a given positive winding number N .

- a) Use the results of the preceding exercise to write the winding number $N = \frac{1}{4\pi} \int n^* \Omega$, and the energy functional $E[\mathbf{n}]$ as

$$4\pi N = \int \frac{4}{(1 + |\xi|^2 + |\eta|^2)^2} (\partial_1 \xi \partial_2 \eta - \partial_1 \eta \partial_2 \xi) dx^1 dx^2,$$

$$E[\mathbf{n}] = \frac{1}{2} \int \frac{4}{(1 + |\xi|^2 + |\eta|^2)^2} ((\partial_1 \xi)^2 + (\partial_2 \xi)^2 + (\partial_1 \eta)^2 + (\partial_2 \eta)^2) dx^1 dx^2,$$

where ξ and η are stereographic co-ordinates on S^2 specifying the direction of the unit vector \mathbf{n} .

- b) Deduce the inequality

$$E - 4\pi N \equiv \frac{1}{2} \int \frac{4}{(1 + |\xi|^2 + |\eta|^2)^2} |(\partial_1 + i\partial_2)(\xi + i\eta)|^2 dx^1 dx^2 > 0.$$

- c) Deduce that for winding number $N > 0$, the minimum energy solutions have energy $E = 4\pi N$ and are obtained by solving the first-order linear equation

$$\left(\frac{\partial}{\partial x^1} + i \frac{\partial}{\partial x^2} \right) (\xi + i\eta) = 0.$$

- d) Solve the equation in part c) and show that the minimal energy solutions with winding number $N > 0$ are given by

$$\xi + i\eta = \lambda \frac{(z - a_1) \dots (z - a_N)}{(z - b_1) \dots (z - b_N)}$$

where $z = x^1 + ix^2$, and λ, a_1, \dots, a_N , and b_1, \dots, b_N , are arbitrary (except for the condition that no a coincides with any b) complex numbers.

- e) Repeat the analysis for $N < 0$. Show that the solutions are given in terms of rational functions of $\bar{z} = x^1 - ix^2$.

The idea of combining the energy functional and the topological charge into a single, manifestly positive, functional is due to Bogomolnyi. The the resulting first order linear equation is therefore called a *Bogomolnyi equation*. If we had tried to find a solution directly in terms of \mathbf{n} , we would have ended up with a horribly non-linear second-order partial differential equation..