Readings/problems 1 for Physics 250

Readings: "Preliminaries" lecture notes; Berry, Proc. Roy. Soc. A (1984); Mermin, RMP 1979, sections I-V. The latter two are available on the internal page.

Exercises: (for 4 units do as many as you can; for 2 units do any one of 1-3 and any one of 4-6.)
1. Generalize our definition of the exterior derivative,

$$df = \sum_j \frac{\partial f_i}{\partial x_j} \, dx_j \wedge dx_i, \tag{1}$$

to forms beyond 1-forms by replacing dx_i by any wedge product of basis forms $dx_i \wedge dx_k \wedge \ldots$. In other words, the derivative of an n-form is obtained by taking the ordinary derivative of the scalar coefficient of each such wedge product and wedge-producting one more form into the wedge product. Show in general that $d^2 = 0$ on any n-form.

2. Discrete Euler characteristic: (a) Check that the cube, octahedron, and tetrahedron have Euler characteristic $V - E + F = 2$, where V is the number of vertices, E of edges, and F of faces. Note that the cube and octahedron are "dual": the vertices of one are the faces of the other. What is the dual of a tetrahedron? (b) (from Fulton) Assume that a polyhedral triangulation of a surface of genus g satisfies $V - E + F = 2 - 2g$. (Here, "triangulation" means that every face is a triangle). Show that $2E = 3F$ and $E \leq V(V-1)/2$ for such a triangulation. Combining this with the assumption, show that

$$V \geq (7 + \sqrt{49 - 24(2 - 2g)})/2. \tag{2}$$

The number on the right side of this inequality is significant: the largest integer less than or equal to $(7 + \sqrt{49 - 24(2 - 2g)})/2$ is the minimal number of colors required to color a map on a surface of this genus (for the sphere, this is the "four-color theorem", which is notoriously difficult to prove).

3. An exact sequence of mappings between groups is one in which the kernel of one mapping (the set of points taken to 0) is equal to the image of the previous mapping. Note that this is a stronger statement than in our sequence of mappings in cohomology theory, where $d^2 = 0$ (as shown in the previous problem) means that the image of one operator d (the exact forms) is a subset of the kernel of the next one (the closed forms). Suppose that the sequence

$$0 \to A \to B \to 0 \tag{3}$$

is exact at both A and B. What can you say about A and B? What if the sequence is only exact at A? at B?

4. Compute the Berry phase obtained by adiabatically rotating the direction of the unit vector $\hat{\mathbf{n}}$ through 2π (i.e., around an axis perpendicular to $\hat{\mathbf{n}}$) in the 2-by-2 Hamiltonian $H = -\alpha \hat{\mathbf{n}} \cdot \mathbf{S}$, where $\mathbf{S}$ means the vector of Pauli matrices and the overall energy scale is α.

5. Consider the path integral for a harmonic oscillator at finite temperature T: with $\beta = 1/k_B T$, the partition function is

$$Z \approx \int dx(\tau) \, e^{-\int_0^\beta \, d\tau \, m\dot{x}^2(\tau)/2 + kx^2/2}, \tag{4}$$

where $x(\tau)$ is required to satisfy periodic boundary conditions: $x(\beta) = x(0)$. Choosing a proper normalization, we want to show that this formula reproduces the known result

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}, \tag{5}$$

with $\omega = \sqrt{k/m}$. Hints: (1) Expand the path $x(\tau)$ in Fourier components, keeping in mind the periodic boundary condition. (2) Use the handy mathematical identity

$$\sinh(z) = z \prod_{k=1}^{\infty} (1 + z^2/(\pi k)^2). \tag{6}$$

(3) Note that the known result above takes a simple form in terms of the sinh function.

6. Consider an electron moving in a ring of radius 1 around a solenoid (directed along the axis of the ring) containing flux Φ. (a) Quantize the one-dimensional Schrödinger equation around the ring. Is there a periodicity in the spectrum with respect to the parameter Φ? (b) Which values of Φ give a spectrum for the electron that is compatible with time-reversal symmetry?