Solutions 1 for Physics 250

1. Generalize our definition of the exterior derivative,

$$df = \sum_j \frac{\partial f_i}{\partial x_j}\, dx_j \wedge dx_i, \tag{1}$$

to forms beyond 1-forms by replacing dx_i by any wedge product of basis forms $dx_i \wedge dx_k \wedge \ldots$. In other words, the derivative of an n-form is obtained by taking the ordinary derivative of the scalar coefficient of each such wedge product and wedge-producting one more form into the wedge product. Show in general that $d^2 = 0$ on any n-form.
Answer: This follows from commutation of partial derivatives and anticommutation of the wedge product. Writing

$$d^2 f = \sum_{j,k} \frac{\partial^2 f_i}{\partial x_j \partial x_k}\, dx_k \wedge dx_j \wedge dx_i, \tag{2}$$

the terms with $j = k$ vanish, and each term with $j \neq k$ is canceled by the term with $k \leftrightarrow j$.

2. Discrete Euler characteristic: (a) Check that the cube, octahedron, and tetrahedron have Euler characteristic $V - E + F = 2$, where V is the number of vertices, E of edges, and F of faces. Note that the cube and octahedron are "dual": the vertices of one are the faces of the other. What is the dual of a tetrahedron? (b) (from Fulton) Assume that a polyhedral triangulation of a surface of genus g satisfies $V - E + F = 2 - 2g$. (Here, "triangulation" means that every face is a triangle). Show that $2E = 3F$ and $E \leq V(V-1)/2$ for such a triangulation. Combining this with the assumption, show that

$$V \geq (7 + \sqrt{49 - 24(2 - 2g)})/2. \tag{3}$$

The number on the right side of this inequality is significant: the largest integer less than or equal to $(7 + \sqrt{49 - 24(2-2g)})/2$ is the minimal number of colors required to color a map on a surface of this genus (for the sphere, this is the "four-color theorem", which is notoriously difficult to prove).
Answer: (a) These are all "spherical" in the sense of having Euler characteristic 2. The tetrahedron has 4 vertices, 4 faces, and 6 edges, and is its own dual. The cube has 8 vertices, 6 faces, and 12 edges, and is dual to the octahedron, which has 6 faces, 8 vertices, and 12 edges.
(b) $2E = 3F$ because, in a triangulation, each edge is shared between two faces, while each face is shared between three edges. $E \leq V(V-1)/2$ because the right-hand-side is the maximum number of distinct edges for a graph of V vertices (i.e., the number of distinct pairs of V elements). Then expressing the Euler characteristic in terms of V leads to a quadratic equation, whose solution is the result.

3. An exact sequence of mappings between groups is one in which the kernel of one mapping (the set of points taken to 0) is equal to the image of the previous mapping. Note that this is a stronger statement than in our sequence of mappings in cohomology theory, where $d^2 = 0$ (as shown in the previous problem) means that the image of one operator d (the exact forms) is a subset of the kernel of the next one (the closed forms). Suppose that the sequence

$$0 \to A \to B \to 0 \tag{4}$$

is exact at both A and B. What can you say about A and B? What if the sequence is only exact at A? at B?
Answer: If the sequence is exact at A, then the set of points in A taken to the identity in B is just the identity, which for a group homomorphism means that the map $A \to B$ is one-to-one. If the sequence is exact at B, then the image of A is equal to B (since all of B is in the kernel), and therefore the map $A \to B$ is onto. If both are satisfied, then the map is an isomorphism and $A \cong B$.

4. Compute the Berry phase obtained by adiabatically rotating the direction of the unit vector $\hat{\mathbf{n}}$ through 2π (i.e., around an axis perpendicular to $\hat{\mathbf{n}}$) in the 2-by-2 Hamiltonian $H = -\alpha \hat{\mathbf{n}} \cdot \mathbf{S}$, where $\mathbf{S}$ means the vector of Pauli matrices and the overall energy scale is α.
Answer: Choose for convenience the spinor gauge

$$|\chi\rangle = \cos(\theta/2)\,|\uparrow\rangle + e^{i\phi}(\sin\theta/2)\,|\downarrow\rangle\,, \tag{5}$$

and let the rotation be around the equator $\theta = \pi/2$. Both the cosine and sine are $1/\sqrt{2}$, and the Berry vector potential is found to be $A_\phi = \partial_\phi(\phi)/2 = 1/2$. The closed loop has length 2π so the Berry phase is π. By spherical symmetry and the gauge-invariance of the Berry phase around a closed loop, this result is generic for any great circle.

5. Consider the path integral for a harmonic oscillator at finite temperature T: with $\beta = 1/k_B T$, the partition function is

$$Z \approx \int dx(\tau)\, e^{-\int_0^\beta d\tau\, m\dot{x}^2(\tau)/2 + kx^2/2}, \tag{6}$$

where $x(\tau)$ is required to satisfy periodic boundary conditions: $x(\beta) = x(0)$. Choosing a proper normalization, we want to show that this formula reproduces the known result

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}, \tag{7}$$

with $\omega = \sqrt{k/m}$. Hints: (1) Expand the path $x(\tau)$ in Fourier components, keeping in mind the periodic boundary condition. (2) Use the handy mathematical identity

$$\sinh(z) = z\prod_{k=1}^{\infty}(1 + z^2/(\pi k)^2). \tag{8}$$

(3) Note that the known result above takes a simple form in terms of the sinh function.
Answer: The continuous-time path integral can be diagonalized by the Fourier transform

$$x(\tau) = \sum_{k=-\infty}^{+\infty} \beta^{-1/2} e^{-2\pi i k t/\beta} y_k. \tag{9}$$

This transforms the action into

$$S[x] = \frac{m}{2}\sum_k \left(\frac{k}{m} + \frac{(2\pi k)^2}{\beta^2}\right)|y_k|^2. \tag{10}$$

Then carrying out the integrals over each y_k and introducing $\omega = \sqrt{k/m}$ gives an answer proportional to (to get the right proportionality constant, one could either discretize the integral carefully)

$$Z \propto \frac{1}{\omega\beta}\prod_{k=1}^{\infty}\left(1 + \frac{\omega^2\beta^2}{(2\pi k)^2}\right)^{-1}. \tag{11}$$

Using the identity provided above gives

$$Z(\beta) = \frac{1}{2\sinh(\omega\beta/2)}, \tag{12}$$

which indeed reproduces the familiar spectrum of the harmonic oscillator. (Getting this functional form, without the overall constant, is OK.)

6. Consider an electron moving in a ring of radius 1 around a solenoid (directed along the axis of the ring) containing flux Φ. (a) Quantize the one-dimensional Schrödinger equation around the ring. Is there a periodicity in the spectrum with respect to the parameter Φ? (b) Which values of Φ give a spectrum for the electron that is compatible with time-reversal symmetry?

Answer: With zero flux, the solution is $E = \hbar^2 k^2/(2m)$ with $k = 2\pi n/2\pi r = n$, for integers n. The spin degree of freedom is independent. With the flux, normalized as $\tilde{\Phi} = \Phi/(\hbar c/e)$, the solution is the same except that the allowed values of k have shifted to $k = (2\pi n + \tilde{\Phi})/2\pi r = n + \tilde{\Phi}/(2\pi)$. So the spectrum is periodic with period $2\pi\hbar c/e = hc/e$, the single-electron flux quantum. (b) Time-reversal symmetry requires that the states k and $-k$ have the same energy (for example, if this is true, one can superpose them to form real wavefunctions). This applies not just at $\Phi = nhc/e$ but also at $\Phi = (n + \frac{1}{2})hc/e$. The second situation is referred to as a "π-flux".