

On the Nonconvexity of the Set of Utility-based Demand Functions

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Abstract

The set of Demand vectors which can be derived from utility maximizing behavior is not *convex*. This implies that the technique of nonparametric regression which seeks to find the closest utility maximizing demand vector to a given set of observations may fail to yield a unique result.

1. Introduction

A fundamental question of economic theory asks "when will a given collection of economics data be consistent with a given theory." For demand analysis, this problem is elegantly solved and elaborated in Afriat (1973) and Varian (1982). The empirical version of this query asks what is the "best" estimate of economic behavior satisfying some theory with respect to a given collection of observations. This estimation problem is customarily addressed through regression analysis but an alternative procedure involving nonparametric analysis has recently experienced renewed interest. This technique, originates with Hildreth (1954) and is described in Varian (1985) and Goldman and Ruud (1993). We wish to address this question of estimating an individual demand function without imposing any restrictions as to functional form except those directly implied by revealed preference. The problem will be described by the attempt to identify a least squares solution to the distance between the actual observations and those obtained from a demand function derivable from utility maximization. In order for a unique

solution to exist to this problem, the allowable set of demand functions in general must be closed and convex. In the note below, we will demonstrate that the set of utility based demand functions fails to be convex and that, consequently, unique estimation is problematic.

2. Basic Notation and Statement of Problem

Suppose that we are faced with N observations of price and income data (P, Y) , and corresponding to these, observations on the bundles demanded X . Let Ξ denote the possible demands which are consistent with utility maximization, i.e.

$$\Xi = \{x | \exists U \text{ s.t. } U(P, Y) = x\}$$

Our problem is then

$$\min_{x \in \Xi} \|x - X\|$$

If $X \in \Xi$ then the observations satisfy the revealed preference axioms and a utility function can be found. Otherwise, we have a violation of the axioms and we are in the position of trying to find the nearest rational demand function to our data. This is, of course, the quadratic programming problem of nonparametric estimation, i.e. find the minimum distance from a point to a set.

The anomaly here lies in the non-convexity of Ξ . The difficulty may be seen by the following rephrasing of the problem: will a convex combination of demand functions, each satisfying the weak axiom of revealed preference, also satisfy the weak axiom? Equivalently, will an aggregate demand function comprised of individual demand functions satisfying the weak axiom, also satisfy that axiom? As is well understood the general answer is negative. The following is an elaboration of an example in Mas-Collel *et al* (1995), Chapter 4C). The description here indicates the general nature of this property.

3. Example

A Walrasian demand function $x(p, y)$ is said to satisfy WARP if $p''x(p', y') \leq y''$ and $x(p'', y'') \neq x(p', y')$ imply $y' < p'x(p'', y'')$.

Consider the following picture:

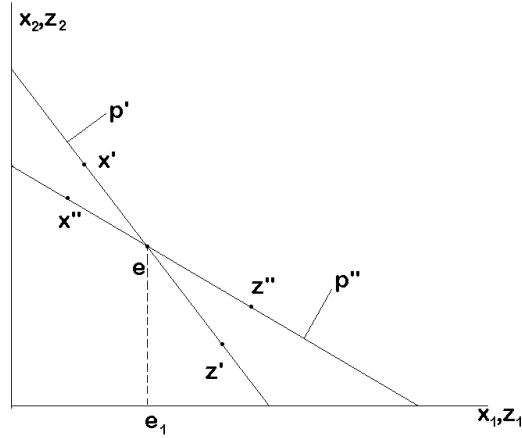


Figure 1: Two Individual's Demand at Two Budgets

x' and x'' represent the choices by an individual which satisfy the weak axiom. Similarly, z' and z'' represent the choices by another individual which also satisfy WARP. In other words, x' and x'' are points on a demand function which satisfy the Weak axiom as are z' and z'' for another demand function. What of the demand function that is a convex combination of the two? Does it satisfy WARP? Equivalently, if a fraction of the population is of type *prime* and the remainder of type *double prime* does the aggregate behavior (or that of the *average* individual) satisfy WARP? Could the average consumer be modelled as utility maximizing?

Consider now the following

Let e denote the intersection of the two budget lines, $e = (e_1, e_2)$.

When $x''_1 \leq e_1$ then $x'_1 \leq e_1$ or else we'd have a violation of WARP. When $x'_1 > e_1$ then $x''_1 \geq e_1$ also to avoid a WARP breach.

If $x''_1 > e_1$ and $x'_1 < e_1$ then there are no WARP implications since neither x' nor x'' are revealed preferred to the other. Thus, demand functions described as having x'_1 and x''_1 in regions I, II or III (in figure 2) will satisfy WARP. But demand functions described as having x'_1 and x''_1 in region IV will fail WARP with $x''_1 < e_1$ and $x'_1 > e_1$. The set of demand functions satisfying WARP is not convex (since there are convex combinations of two

WARP satisfying bundles in regions I and III which fall in region IV).

For example, if 50% of the population were at α (in region I) and 50% at β (in region III), the average member (i.e. the aggregate demand divided by the total population) would fall at γ (in region IV) and would violate WARP.

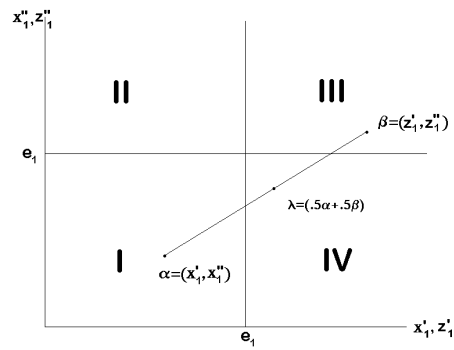


Figure 2: A Reparametrization of the Demand Space

Referring back to the figure 1, the average of x' and z' is to the right of e along the *prime* budget line and that of x'' and z'' is to the left of e along the *double prime* budget line, a clear violation of WARP. This result is illustrated in figure 3 below.

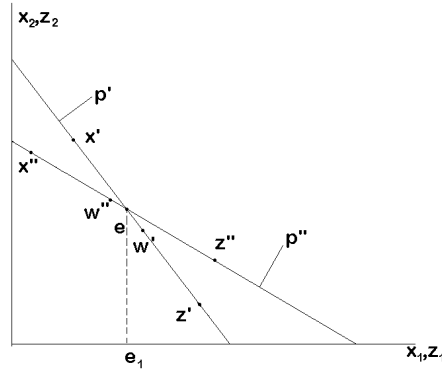


Figure 3: The Convex Combination Violates WARP

4. Bibliography

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